

Long velocity tails in plasmas and gravitational systems

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Abstract

Long tails in the velocity distribution are observed in plasmas and gravitational systems. Some experiments and observations in far-from-equilibrium conditions show that these tails behave as $1/v^{5/2}$. We show here that such heavy tails are due to a universal mechanism related to the fluctuations of the total force field. Owing to the divergence in $1/r^2$ of the binary interaction force, these fluctuations can be very large and their probability density exhibits a similar long tail. They induce large velocity fluctuations leading to the $1/v^{5/2}$ tail. We extract the mechanism causing these properties from the BBGKY hierarchy representation of Statistical Mechanics. This leads to a modification of the Vlasov equation by an additional term. The novel term involves a fractional power $3/4$ of the Laplacian in velocity space and a fractional iterated time integral. Solving the new kinetic equation for a uniform system, we retrieve the observed $1/v^{5/2}$ tail for the velocity distribution. These results are confirmed by molecular dynamics simulations.

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I. INTRODUCTION

Long tails, i.e. asymptotic behaviour in $1/v^\alpha$, $\alpha > 0$, for large velocity v of the velocity distribution are frequently observed or inferred in experiments with far-from-equilibrium plasmas. Similar tails are also obtained from astronomical data about large-scale galaxy systems. More precisely, in experiments with focused ion beams [1], the observed transverse velocity distribution of the ions is a symmetric Lévy-stable distribution of stability index $3/2$ [2] with a long tail in $1/v^{5/2}$ for any component of the transverse velocity vector. Other processes implying long tails in the velocity distribution are the phenomena of ionisation and nuclear fusion in plasmas. These processes are very sensitive to the number of high velocity particles in the system. In these experiments the measured rates are often significantly larger than those predicted using a Gaussian velocity distribution [3]. Such discrepancies led some researchers to replace the Gaussian velocity distribution by a convolution between a Gaussian and a symmetric Lévy of index $3/2$ [4]. The rates calculated with this new distribution came closer to the measured ones [3].

A qualitative explanation is the following. Consider a large system of particles interacting via the Coulomb or the gravitational force and in which spatial correlations between particles can be neglected. The distribution of the random total force field \vec{F} due to all the particles is a Lévy- $3/2$ distribution, also called Holtsmark distribution [5] [6]. The reduced Holtsmark distribution for any component F_i of the 3-dimensional vector \vec{F} has a long tail in $1/F_i^{5/2}$. This tail denotes a much greater probability for large force fluctuations than with a Gaussian distribution. It originates from the divergence at short distances of the $1/r^2$ interaction force. Yet, for short enough times, the velocity of a particle submitted to the total force \vec{F} is essentially proportional to that field. Hence, for such short times, the distribution of the velocities should be a convolution of a Lévy- $3/2$ distribution with the initial

velocity distribution. For initial distributions with finite second order moments, this convolution gives a heavy tail in $1/v^{5/2}$ [7].

Similar results are obtained in large-scale systems of galaxies for the distribution of the peculiar velocities of the galaxies [8]. The peculiar velocity is the difference between the observed velocity and the local cosmological expansion velocity. The observed peculiar velocity distribution has a long tail in $1/v^{2.1}$. The discrepancy between the exponent 2.1 of the observed velocity tail and the exponent 2.5 of the tail of a true Holtsmark distribution is explained by the fractal structure of the spatial distribution [9] [10].

The Vlasov equation on which the current theories of plasmas and gravitational systems mainly rely involves only the effects of the average total force field. Since it excludes fluctuations around that average, this equation cannot have Lévy distributions as generic solutions. A theoretical explanation is thus required. In the present work we derive from the BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) hierarchy of equations [11] a modified kinetic equation for systems interacting through forces in $1/r^2$. It differs from the Vlasov equation by a new additive contribution. That term involves a fractional power $3/4$ of the Laplacian operator in velocity space and a fractional iterated integral in time. The kinetic equation is derived in chapter 1. In chapter 2, its solution is obtained for a spatially uniform system and is shown to possess the observed $1/v^{5/2}$ tail. In chapter 3, we present molecular dynamics simulations that confirm this result. Conclusions and perspectives are discussed in the last chapter.

II. DERIVING THE KINETIC EQUATION FROM THE BBGKY HIERARCHY

We consider a system of N identical classical point-like particles of mass m in \mathbb{R}^3 . They interact via a binary potential $U(\vec{r}) = \gamma/r$. The variable r is the norm of the distance vector \vec{r} between the two interacting particles. In order to cover both repulsive and attractive interactions in charged particles gases and gravitational systems, the coupling constant γ can be positive or negative.

In classical Statistical Mechanics a system is described by the 1-particle phase-space distribution function (1-pdf) $f_1(\vec{r}_1, \vec{v}_1; t) \equiv f(\mathbf{1}; t)$ and by the phase-space correlation functions of growing orders such as $g_2(\mathbf{1}, \mathbf{2}; t) \equiv f_2(\mathbf{1}, \mathbf{2}; t) - f(\mathbf{1}; t)f(\mathbf{2}; t)$ where $f_2(\mathbf{1}, \mathbf{2}; t)$ is the 2-particles phase-space distribution, g_3 and so on [11]. Here, the index $\mathbf{i} = \mathbf{1}, \mathbf{2}, \dots, \mathbf{N}$ denotes the set of position and velocity variables, \vec{r}_i, \vec{v}_i , of particle i in the system. The time-evolution of the 1-pdf f and the phase-space correlations g_2, g_3, \dots obey the so-called BBGKY hierarchy of coupled equations [11]. Our aim is to derive from that hierarchy a kinetic equation, that is, a closed equation for the 1-pdf of the system. This can only be achieved by a truncation of the hierarchy. This truncation must, of course, be based on some specific properties of the system. In this chapter we present the main steps leading to the kinetic equation for plasmas and gravitational systems. More details are given in the Methods A section.

The first equation of the BBGKY hierarchy [11] reads,

$$\partial_t f(\mathbf{1}; t) = L_1^0 f(\mathbf{1}; t) + \int d\mathbf{2} L'_{12} f(\mathbf{1}; t) f(\mathbf{2}; t) + \int d\mathbf{2} L'_{12} g_2(\mathbf{1}, \mathbf{2}; t) \quad (1)$$

The free motion operator L_i^0 corresponds to $L_i^0 \equiv -\vec{v}_i \cdot \frac{\partial}{\partial \vec{r}_i}$. The interaction operator L'_{ij} is given by $L'_{ij} \equiv -\frac{1}{m} \vec{F}(\vec{r}_i - \vec{r}_j) \cdot (\frac{\partial}{\partial \vec{v}_i} - \frac{\partial}{\partial \vec{v}_j})$ where $\vec{F}(\vec{r}_i - \vec{r}_j) \equiv \vec{F}_{ij} \equiv \gamma \frac{\vec{r}_i - \vec{r}_j}{\|\vec{r}_i - \vec{r}_j\|^3}$ is the interaction force of particle i acting on j . The integral symbol $\int d\mathbf{i}$ stands for

$\int d^3r_i \int d^3v_i$ where both integrals are over \mathbb{R}^3 . The integration domain over \vec{r}_i should be the volume V of the system, but in view of the thermodynamic limit, $N \rightarrow \infty$, $V \rightarrow \infty$, $N/V = n = \text{constant} < \infty$, considered here, the domain is assimilated to \mathbb{R}^3 . The thermodynamic limit should not be confused with the fluid limit in which $N \rightarrow \infty$, $m \rightarrow 0$, $\gamma \rightarrow 0$, $Nm = \text{constant} < \infty$, $N\gamma = \text{constant} < \infty$. The fluid limit removes the discrete character of the particles. Since we are interested in a phenomenon related to the discreteness of particles, the thermodynamic limit is taken here. For a discussion of the two limits see reference [12].

In the right-hand-side of equation (1), the first term represents free motion while the second term denotes the Vlasov term. The latter describes the effect of the mean force field due to all the other particles on particle **1** [11]. This field is the average of the force over the 1-pdf itself. It vanishes for uniform systems (see discussion in Methods A.1 after equation (24)). The third term in the right-hand-side of equation (1) takes into account the 2-particles phase-space correlations g_2 and couples this equation to the rest of the BBGKY hierarchy as we see below.

Our concern is limited to weakly coupled systems, i.e. systems for which $\Gamma \equiv U/kT \ll 1$. Here, kT denotes the average kinetic energy of two particles. The quantity $U \equiv |U(\delta)| = |\gamma|/\delta$ represents the potential energy between two particles at the average distance $\delta = n^{-1/3}$ between nearest neighbors. The weak coupling condition is, thus, $\Gamma = |\gamma| n^{1/3}/kT \ll 1$. We also limit our scope to short time evolutions such that $t \ll t_r$ where t_r is the relaxation time to equilibrium. For weakly coupled systems, in the current theories, the third term $C \equiv \int d\mathbf{2} L'_{12} g_2(\mathbf{1}, \mathbf{2}; t)$ in equation (1) is shown to contain the effect of binary collisions [11]. Under these collisions the system irreversibly relaxes towards equilibrium in a time $t_r \sim t_s \Gamma^{-3/2}$ [13], where $t_s = (\frac{m}{4\pi|\gamma|n})^{1/2}$. The short time-scale t_s represents for plasmas, the plasma oscillations period and for gravitational systems, the free fall time. Since $\Gamma \ll 1$, t_r is a

very large time. Hence, for times $t \ll t_r$ such as we consider, the term C can be neglected. This reduces equation (1) to the well-known Vlasov equation.

However, for interaction forces that diverge as $1/r^2$ at small distances the above reasoning does not hold. Statistically, the divergence increases the probability of large fluctuations of the force which, in turn, results in the long tail of the total force distribution. This effect should be found in C as the integral over \vec{r}_2 that it involves contains the near vicinity of the origin where the divergence of the force occurs. That part of C , thus, cannot a priori be neglected. To explicit this remark, we divide the integration domain of the integral over \vec{r}_2 in C in two parts: a small open ball S_1 of radius d centered at particle **1**, and the rest of the space, $\mathbb{R}^3 \setminus S_1$. The radius d is defined such that the average interaction energy between any particle 2 located in that sphere and particle **1** is larger than the sum of the average kinetic energies of these two particles. We, thus, have $\frac{|\gamma|/d}{kT} = 1$ which amounts to $d = |\gamma|/kT$. We also assume that the typical macroscopic inhomogeneity length, L_H , is much larger than d , $d \ll L_H$ and also that $\delta \ll L_H$. Combined with $\Gamma \ll 1$, this yields $d \ll \delta \ll L_H$.

We, thus, get

$$C = I_1 + I_2 \quad (2)$$

with,

$$I_1 = \int_{S_1} d^3 r_2 \int d^3 v_2 L'_{12} g_2(\vec{r}_1, \vec{v}_1, \vec{r}_2, \vec{v}_2; t) \quad (3)$$

and,

$$I_2 = \int_{\mathbb{R}^3 \setminus S_1} d^3 r_2 \int d^3 v_2 L'_{12} g_2(\vec{r}_1, \vec{v}_1, \vec{r}_2, \vec{v}_2; t) \quad (4)$$

Owing to the above splitting, the norm of the interaction force $F_{12} \equiv \|\vec{F}_{12}\|$ is large in I_1 while it is small in I_2 . As already mentioned, the latter is known to lead

to the collision term in the Landau or the Balescu-Lenard equations [11]. Since our system is assumed to be weakly coupled and, more importantly, as we are interested in times $t \ll t_r$, the term I_2 can safely be neglected. However, in view of what has been discussed above, by no means can we neglect I_1 as is usually done. Equation (1), hence, becomes

$$\begin{aligned} \partial_t f(\mathbf{1}; t) = & L_1^0 f(\mathbf{1}; t) + \int d\mathbf{2} L'_{12} f(\mathbf{1}; t) f(\mathbf{2}; t) \\ & + \int_{S_1} d^3 r_2 \int d^3 v_2 L'_{12} g_2(\vec{r}_1, \vec{v}_1, \vec{r}_2, \vec{v}_2; t) \end{aligned} \quad (5)$$

Let us stress that in the standard derivations of the kinetic equations [11] the discussion of I_1 is eluded. It is simply replaced by a cut-off at short distances in the integral over r_2 in C in order to avoid that divergence. Usually, this cut-off is justified by assuming an effective quantum repulsion at short distances. This means that in these theories C , in fact, reduces to I_2 . Consequently, in these theories, for short time $t \ll t_r$ during which the effects of collisions are negligible, the only remaining terms are the free motion and the Vlasov mean-field terms. In other words, one gets the Vlasov equation. In contrast, in the sequel we show that I_1 does not diverge (see Methods A.2 after equation (32)) and the Vlasov equation must be modified by this term. We now proceed to calculate I_1 .

A glance at equation (5) shows that it is not closed. It is coupled to the second equation of the BBGKY hierarchy [11] via the unknown function $g_2(\vec{r}_1, \vec{v}_1; \vec{r}_2, \vec{v}_2; t)$. Equation (5) is, thus, coupled to the second equation of the BBGKY hierarchy [11]

$$\begin{aligned} \partial_t g_2(\mathbf{1}, \mathbf{2}; t) = & [L_1^0 + L_2^0] g_2(\mathbf{1}, \mathbf{2}; t) + L'_{12} [g_2(\mathbf{1}, \mathbf{2}; t) + f(\mathbf{1}; t) f(\mathbf{2}; t)] \\ & + \int d\mathbf{3} \{ L'_{13} f(\mathbf{1}; t) g_2(\mathbf{2}, \mathbf{3}; t) + L'_{23} f(\mathbf{2}; t) g_2(\mathbf{1}, \mathbf{3}; t) \\ & + (L'_{13} + L'_{23}) [f(\mathbf{3}; t) g_2(\mathbf{1}, \mathbf{2}; t) + g_3(\mathbf{1}, \mathbf{2}, \mathbf{3}; t)] \} \end{aligned} \quad (6)$$

which must be considered with the constraint $\|\vec{r}_2 - \vec{r}_1\| < d$.

Obviously, this equation is also not closed. It is coupled to the third BBGKY equation -not written here- via the 3-particles phase-space correlation g_3 . The latter is coupled to the equation for g_4 and so on, generating the whole BBGKY hierarchy of coupled equations [11]. However, as shown in Methods A.1, the conditions $\|\vec{r}_2 - \vec{r}_1\| < d$, $t \ll t_r$ along with $d \ll \delta \ll L_H$ allow for truncating and greatly simplifying this hierarchy. Indeed, it turns out that equation (6) reduces to the closed equation,

$$\partial_t g_2(\mathbf{1}, \mathbf{2}; t) = L'_{12} [g_2(\mathbf{1}, \mathbf{2}; t) + f(\mathbf{1}; t) f(\mathbf{2}; t)] \quad (7)$$

As shown in Methods A.2, the solution of this equation is easily found and is introduced in the expression (3) of I_1 , yielding,

$$I_1 \approx -\frac{1}{5} \left(\frac{2\pi\gamma}{m} \right)^{3/2} \int_0^t \frac{d\tau}{\sqrt{\tau}} n(\vec{r}_1; t - \tau) (-\Delta_{\vec{v}_1})^{3/4} f(\vec{r}_1, \vec{v}_1; t - \tau) \quad (8)$$

where $n(\vec{r}_1; t) \equiv \int d^3v f(\vec{r}_1, \vec{v}; t)$ is the local number density. The fractional power $3/4$ of the Laplacian operator in the velocity variable is defined by

$$(-\Delta_{\vec{v}_1})^{3/4} e^{i\vec{\zeta}_1 \cdot \vec{v}_1} \equiv (\vec{\zeta}_1 \cdot \vec{\zeta}_1)^{3/4} e^{i\vec{\zeta}_1 \cdot \vec{v}_1} = \zeta_1^{3/2} e^{i\vec{\zeta}_1 \cdot \vec{v}_1} \quad (9)$$

and by using the Fourier integral representation of $f(\vec{r}_1, \vec{v}_1; t - \tau)$ with respect to the velocity.

Finally, replacing expression (8) into equation (5), a closed equation for the 1-pdf is obtained. This is the final form of our kinetic equation

$$\begin{aligned}
\partial_t f(\vec{r}_1, \vec{v}_1; t) &= -\vec{v}_1 \cdot \vec{\nabla}_1 f(\vec{r}_1, \vec{v}_1; t) \\
&- \frac{1}{m} \int d^3 r_2 \int d^3 v_2 \vec{F}_{12} \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) f(\vec{r}_1, \vec{v}_1; t) f(\vec{r}_2, \vec{v}_2; t) \\
&- \frac{1}{5} \left(\frac{2\pi\gamma}{m} \right)^{3/2} \int_0^t \frac{d\tau}{\sqrt{\tau}} n(\vec{r}_1; t - \tau) (-\Delta_{\vec{v}_1})^{3/4} f(\vec{r}_1, \vec{v}_1; t - \tau)
\end{aligned} \tag{10}$$

Equation (10) is a Vlasov equation modified by the addition of a new term. Like the Vlasov term, the new contribution is nonlinear in the 1-pdf due to the presence in it of $n(\vec{r}_1; t)$. Notice that in solving equation (7) the initial correlation has been chosen to vanish (see Methods A.2). As discussed in chapter 5, a non-vanishing initial correlation would not change our main conclusions.

We next show that, in the case of homogeneous systems, the general solution to the kinetic equation is a Lévy-3/2 distribution with a long tail in $1/v^{5/2}$.

III. HOMOGENEOUS SYSTEMS

We now consider the particular situation of a spatially uniform system. For such systems the free motion term as well as the Vlasov term exactly vanish in the kinetic equation (10). This leaves only the new term in that equation and singles out its effect on the evolution of the 1-pdf.

For uniform systems the 1-pdf becomes $f(\vec{r}, \vec{v}; t) = n\varphi(\vec{v}; t)$, where $\varphi(\vec{v}; t)$ is the velocity distribution at time t . The equation (10) then becomes

$$\partial_t \varphi(\vec{v}; t) = -\frac{n}{5} \left(\frac{2\pi\gamma}{m} \right)^{3/2} \int_0^t \frac{d\tau}{\sqrt{\tau}} (-\Delta_v)^{3/4} \varphi(\vec{v}; t - \tau) \tag{11}$$

Obviously, the kinetic equation becomes exactly linear. This offers the possibility to find the exact solution of this equation. Such is not the case for the general

nonlinear equation (10). The solution of equation (11) is easily found (see Methods B). For short times, it reduces to a velocity-convolution between the initial velocity distribution and a Lévy-3/2 distribution [2],

$$\varphi(\vec{v}; t) \simeq \int d^3u \varphi(\vec{u}; 0) L_{3/2}(\vec{v} - \vec{u}, Ct^{3/2}) \quad (12)$$

with $C = \frac{4n}{15} \left(\frac{2\pi\gamma}{m} \right)^{3/2}$ and where $L_{3/2}(\vec{v}, Ct^{3/2})$ denotes the probability density of a multivariate isotropic Lévy-stable random variable \vec{v} centered at zero with, in the notations of J.P.Nolan [2], stability index $\alpha = 3/2$ and scale factor $\tilde{\gamma} = C^{2/3}t$.

For all initial distributions with finite second moments, the above convolution gives a long-tailed distribution with tail $1/v^{5/2}$ [7] where v is any component of the velocity vector \vec{v} . This result alone justifies the derivation of the new kinetic equation (10) as it establishes a clear connection with experimental and numerical results as we see in next chapter.

At this level, one could question the physical soundness of such a distribution. Indeed, its second moment and all its higher order moments diverge. Consequently, the average kinetic energy obtained from it diverges. This divergence is due to the fact that particles that are very near each other acquire very large accelerations and velocities under the action of the divergent interaction force. However, this never occurs in reality as natural cut-offs appear at very short distances. For elementary particles they originate from quantum effective repulsion and, for macroscopic bodies, from internal cohesion forces that maintain their shape. This results in a form of regularisation of the potential (see next chapter) and leads to a natural truncation of the tails of these distributions. That truncation ensures the existence of the second and higher order moments. Nevertheless, the long tail property essentially persists, modified at only extremely large values of the velocity by a sharp decrease.

A last result is worth to be reported. Equation (11) can be cast into a particularly

elegant form. The fractional iterated integral operator of order $1/2$ acting on a function $f(t)$ is defined [15] by $J_t^{1/2}f(t) \equiv \frac{1}{\sqrt{\pi}} \int_0^t \frac{d\tau}{\sqrt{\tau}} f(t - \tau)$. Up to a factor $1/\sqrt{\pi}$ this is just the integral operator on variable τ appearing in the right hand side of equation (11). The Riemann-Liouville fractional derivative of order $1/2$ in the time variable t is defined [15] as: $D_t^{1/2} \equiv \frac{d}{dt} \circ J_t^{1/2}$. Let us apply $D_t^{1/2}$ on both sides of equation (11). Using the following properties and definitions [15], $J_t^{1/2} \circ J_t^{1/2} = J_t^1$, $\frac{d}{dt} \circ J_t^1 = I$, $\frac{d}{dt} \circ D_t^{1/2} \equiv D_t^{3/2}$, where I is the identity operator, the equation (11) transforms into

$$D_t^{3/2}\varphi(\vec{v};t) = -A(-\Delta_{\vec{v}_1})^{3/4} \varphi(\vec{v};t) \quad (13)$$

where $A \equiv \frac{3}{4}\pi^{1/2}C$. To our knowledge, this is the first time such a purely fractional partial differential equation is derived from the basic principles of Statistical Mechanics.

IV. NUMERICAL SIMULATIONS

We simulated a 3D gravitational system of 131,072 identical point-like classical particles using a fourth order symplectic integrator. The molecular dynamical code was implemented in the CUDA parallel computing architecture on graphic processing units [14]. The initial distribution of particles is spatially uniform in a spherical volume with all the particles at rest and without space boundaries: The system is open and particles can escape. The interaction potential $\frac{\gamma}{r}$ is regularized into $\frac{\gamma}{(r^2+\varepsilon^2)^{1/2}}$ in order to avoid divergences in the numerical integration. The statistical significance is increased by performing 100 realisations (runs).

After a very short time, long tails proportional to $1/v^\alpha$ develop in the distribution for any component v of the velocity vector \vec{v} (see Figures 1 a and 1b).

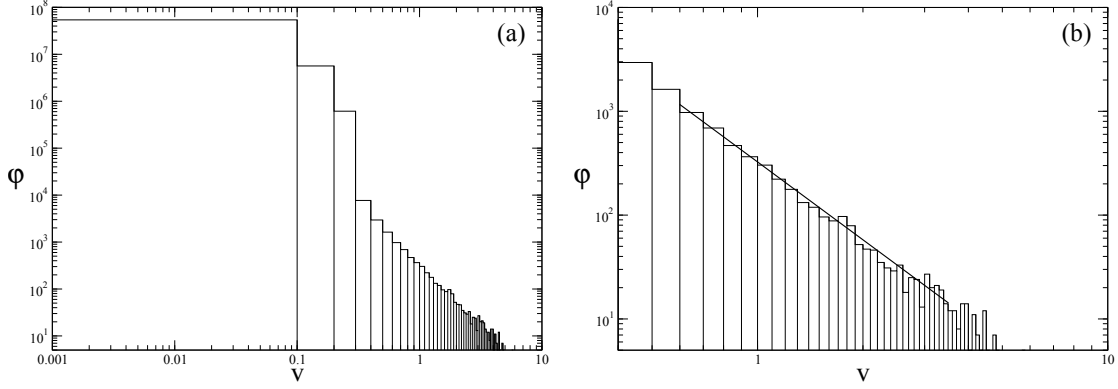


Figure 1. (a) Log-log histogram of distribution of any component v of velocity \vec{v} for regularisation parameter $\varepsilon = 10^{-16}$. Total number of particles $N = 131,072$. Number of realisations = 100. Initial velocities = 0.00. Initial spatial distribution: uniform in a sphere of radius $R = 1.28$. Time step = $2 \times 10^{-7}T$, total run time = $3 \times 10^{-6}T$ with $T = (nGm)^{-1/2}$, G is the gravitational constant.

(b) Zoom on the tail of the velocity distribution given in (a). The thick straight line is the result of a linear regression on the tail with slope = -2.49 , standard error = 0.13 and correlation coefficient = -0.973 .

Figure 2 shows the exponent α for decreasing values of ε and same initial conditions. Each point corresponds to 100 runs (realisations). As observed, α approaches the theoretical value $5/2$ as ε gets smaller.

The tail of the distribution is very sensitive to the behavior of the interaction force at small distances and, consequently, to ε , such that the expected behaviour is observed only for very small values of ε . Note that the errors (from 5% to 10%) on α should not appear as a great concern as they could be greatly reduced by increasing either the number of particles in the simulation or the number of realisations used for statistical significance. Indeed, the tail of a distribution is much more sensitive to the finiteness of N than its bulk. The reason is that the $1/v^\alpha$ tail of a Lévy distribution

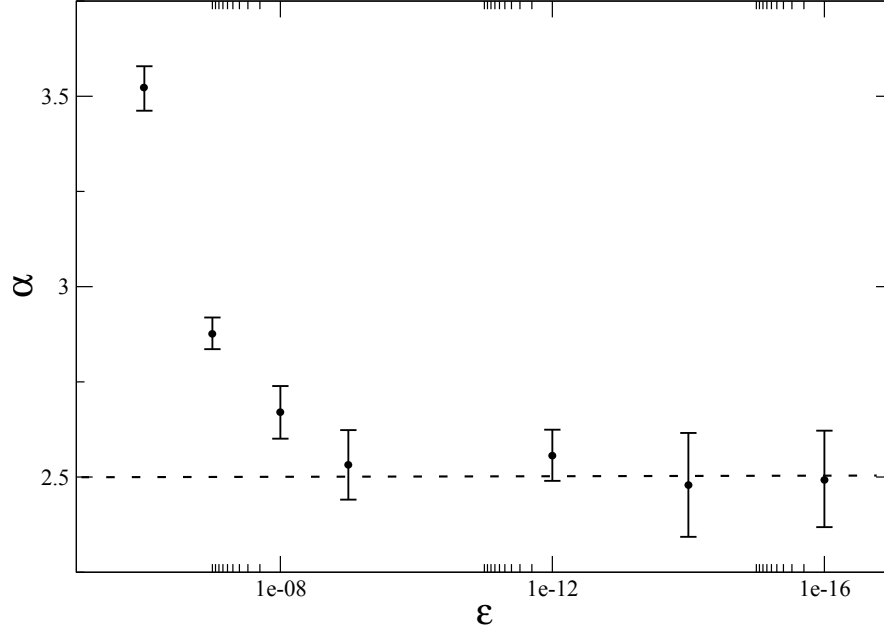


Figure 2. Semi-log graph of the tail exponent α in $1/v^\alpha$ where v is one component of velocity \vec{v} , as a function of the regularisation parameter ϵ . Each point is obtained from a linear regression on velocity data obtained from 100 realisations (runs) as in Figure 1.b. Error bars correspond to the standard deviation obtained from the linear regression and the dashed line is the theoretical value at $\alpha = 2.5$.

is a consequence of a generalized Central-Limit theorem [2] which only holds at the limit $N \rightarrow \infty$. Yet, numerical simulations (and real systems too) are bound to involve a finite number of particles. As a consequence, the number of particles with very large values of the velocity, i.e. those that constitute the tail of the distribution, is finite and represents only a small fraction of N . The shape of the tail, thence, is subject to large fluctuations.

V. PERSPECTIVES AND CONCLUSIONS

The new term in the kinetic equation (10) is of order $1/N$ compared to the Vlasov term. However, in uniform or near-uniform spatial configurations, while the Vlasov term vanishes or is very small, this term remains finite and, consequently, cannot be neglected. In fully inhomogeneous states, furthermore, though small compared to the Vlasov term, this new term might have some important consequences. Indeed, different finite samples of N points obeying a statistical distribution with an algebraic tail usually have very different standard deviations (diverging with N). As a consequence, this new term may have a measurable influence on the quasi-stationary state that appears after the violent relaxation process [16] [17]. Also, its magnitude is of the same order as the collisional corrections to the Vlasov equation. The latter become important for long times. Therefore, an extension of the present approach to the relaxation time-scale may be of interest. Longer simulation runs are required to study how the long tails disappear and how the new term affects the evolution of the system on longer time scales.

In our derivation of the kinetic equation (10) we assumed a vanishing initial binary correlation function (see Methods A.2). What would be the effect of a non-vanishing initial correlation? We studied this question and, briefly, the results are the following. Non-vanishing initial correlations introduce a source term in the kinetic equation. The solution of that equation involves a time convolution between this source term and the propagator of equation (10). Using a theorem in reference [7], one then shows for uniform systems that the tail of the resulting distribution remains proportional to $1/v^{5/2}$ for a large class of initial correlations.

The present theory extends without difficulty to systems with more general interaction forces that behave as $1/r^2$ only at short distances as, for instance, systems

interacting via a Yukawa-type or Debye-screened potential $\gamma \frac{e^{-r/\lambda}}{r}$. Moreover, if in addition one has $\lambda \ll L_H$, i.e. the interaction is short-ranged, then the Vlasov term is negligible [11] and the supplementary term we derived is dominant in the kinetic equation. Yukawa-like effective potentials also play an important role in nuclear physics and, more particularly, in heavy-ion collisions such as those occurring in the large accelerators. However, the nuclear effective interaction is more complex than the Yukawa potential[18]. The former depends on the spins and isospins of the two interacting particles. In some spin and isospin states the potential diverges as $1/r$ at short distances, in others it behaves as $1/r^3$. Quantum effects are, thus, important in heavy-ion interactions and would require a quantum or, at least, a semi-classical extension of our approach.

VI. METHODS

A. Derivation of the kinetic equation (10)

1. Truncation of the BBGKY hierarchy for $\|\vec{r}_2 - \vec{r}_1\| < d$

Let us analyse the integral term in equation (6). We denote it by K

$$K = \int_{\mathbb{R}^3} d^3r_3 \int_{\mathbb{R}^3} d^3v_3 \left\{ L'_{13} f(\mathbf{1}; t) g_2(\mathbf{2}, \mathbf{3}; t) + L'_{23} f(\mathbf{2}; t) g_2(\mathbf{1}, \mathbf{3}; t) + (L'_{13} + L'_{23}) [f(\mathbf{3}; t) g_2(\mathbf{1}, \mathbf{2}; t) + g_3(\mathbf{1}, \mathbf{2}, \mathbf{3}; t)] \right\} \quad (14)$$

K is the sum of four contributions $K = K_1 + K_2 + K_3 + K_4$ that are defined below, one after the other. The first one is

$$K_1 = \int_{\mathbb{R}^3} d^3r_3 \int_{\mathbb{R}^3} d^3v_3 L'_{13} f(\mathbf{1}; t) g_2(\mathbf{2}, \mathbf{3}; t) \quad (15)$$

or more explicitly and with a permutation of integrals

$$K_1 = -\frac{1}{m} \int_{\mathbb{R}^3} d^3 v_3 \int_{\mathbb{R}^3} d^3 r_3 \vec{F}(\vec{r}_1 - \vec{r}_3) \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_3} \right) f(\vec{r}_1, \vec{v}_1; t) g_2(\vec{r}_2, \vec{v}_2, \vec{r}_3, \vec{v}_3; t) \quad (16)$$

where $\vec{F}(\vec{r}) \equiv \gamma \frac{\vec{r}}{r^3}$. The part of the volume integral over \vec{v}_3 containing $\frac{\partial}{\partial \vec{v}_3}$ transforms into a surface integral on the surface at infinity in the sub-space of velocity \vec{v}_3 and vanishes due to the fact that $g_2(\vec{r}_2, \vec{v}_2, \vec{r}_3, \vec{v}_3; t) \rightarrow 0$ for $v_3 \rightarrow \infty$ [11]. Let us make successively two changes of variable in the integral over \vec{r}_3 : First, $\vec{r}_3 \rightarrow \vec{r} = (\vec{r}_1 - \vec{r}_3)$ and, second, $\vec{r} \rightarrow \vec{F} = \gamma \frac{\vec{r}}{r^3}$. In the last transformation the volume element becomes $d^3 r = \frac{1}{2} \gamma^{3/2} F^{-9/2} d^3 F$. Hence, K_1 reads now

$$K_1 = -\frac{\gamma^{3/2}}{2m} \int_{\mathbb{R}^3} d^3 v_3 \int_{\mathbb{R}^3} d^3 F F^{-9/2} \vec{F} g_2(\vec{r}_2, \vec{v}_2, \vec{r}_1 - \gamma^{1/2} F^{-3/2} \vec{F}, \vec{v}_3; t) \cdot \frac{\partial}{\partial \vec{v}_1} f(\vec{r}_1, \vec{v}_1; t) \quad (17)$$

We now express the integral over \vec{F} in spherical coordinates F, θ, φ

$$K_1 = -\frac{\gamma^{3/2}}{2m} \int_{\mathbb{R}^3} d^3 v_3 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi \vec{n}(\theta, \varphi) \int_0^\infty dF F^{-3/2} g_2(\vec{r}_2, \vec{v}_2, \vec{r}_1 - \gamma^{1/2} F^{-1/2} \vec{n}(\theta, \varphi), \vec{v}_3; t) \cdot \frac{\partial}{\partial \vec{v}_1} f(\vec{r}_1, \vec{v}_1; t) \quad (18)$$

where $\vec{n}(\theta, \varphi)$ is the unit vector

$$\vec{n}(\theta, \varphi) = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}$$

Finally, the change of variable $F \rightarrow u = F^{-1/2}$ yields

$$K_1 = -\frac{\gamma^{3/2}}{m} \int_{\mathbb{R}^3} d^3 v_3 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi \vec{n}(\theta, \varphi) \int_0^\infty du g_2(\vec{r}_2, \vec{v}_2, \vec{r}_1 - \gamma^{1/2} u \vec{n}(\theta, \varphi), \vec{v}_3; t) \cdot \frac{\partial}{\partial \vec{v}_1} f(\vec{r}_1, \vec{v}_1; t) \quad (19)$$

Notice that the divergence of the integral over \vec{r}_3 in equation (16) that could have been expected from the divergence of the force when $\vec{r}_3 \rightarrow \vec{r}_1$ does not occur here.

Indeed, in its transformed form (19) the integral over u contains only g_2 which, in turn, must be an integrable function of all its arguments. The last claim comes from the fact that the two-particles phase-space distribution must be integrable in order to be normalised. Hence, for all values of \vec{r}_1 and \vec{r}_2 the term K_1 is finite. This contrasts with the term $\mathcal{L} \equiv L'_{12}[g_2(\mathbf{1}, \mathbf{2}; t) + f(\mathbf{1}; t) f(\mathbf{2}; t)]$ of equation (6) where in L'_{12} the force diverges for $\vec{r}_2 \rightarrow \vec{r}_1$. Since equation (6) is considered here with the constraint $\|\vec{r}_2 - \vec{r}_1\| < d$, \mathcal{L} is dominant over K_1 . The same argument applies to K_2 with the permutation $1 \longleftrightarrow 2$

$$K_2 = \int_{\mathbb{R}^3} d^3 r_3 \int_{\mathbb{R}^3} d^3 v_3 L_{23} f(\mathbf{2}; t) g_2(\mathbf{1}, \mathbf{3}; t) \quad (20)$$

Using the same changes of variables as above, the term K_3

$$K_3 = \int_{\mathbb{R}^3} d^3 r_3 \int_{\mathbb{R}^3} d^3 v_3 (L'_{13} + L'_{23}) g_3(\mathbf{1}, \mathbf{2}, \mathbf{3}; t) \quad (21)$$

becomes

$$\begin{aligned} K_3 = & -\frac{\gamma^{3/2}}{m} \frac{\partial}{\partial \vec{v}_1} \cdot \int_{\mathbb{R}^3} d^3 v_3 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi \vec{n}(\theta, \varphi) \int_0^\infty du g_3(\vec{r}_1, \vec{v}_1, \vec{r}_2, \vec{v}_2, \vec{r}_1 - \gamma^{1/2} u \vec{n}(\theta, \varphi), \vec{v}_3; t) \\ & - \frac{\gamma^{3/2}}{m} \frac{\partial}{\partial \vec{v}_2} \cdot \int_{\mathbb{R}^3} d^3 v_3 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi \vec{n}(\theta, \varphi) \int_0^\infty du g_3(\vec{r}_1, \vec{v}_1, \vec{r}_2, \vec{v}_2, \vec{r}_2 - \gamma^{1/2} u \vec{n}(\theta, \varphi), \vec{v}_3; t) \end{aligned} \quad (22)$$

and with a similar argument as for K_1 and K_2 one can neglect K_3 with respect to \mathcal{L} . Finally, let us consider the term K_4

$$K_4 = \int_{\mathbb{R}^3} d^3 r_3 \int_{\mathbb{R}^3} d^3 v_3 (L'_{13} + L'_{23}) f(\mathbf{3}; t) g_2(\mathbf{1}, \mathbf{2}; t) \quad (23)$$

More explicitly

$$K_4 = -\frac{1}{m} \left\{ \int_{\mathbb{R}^3} d^3 r_3 \vec{F}(\vec{r}_1 - \vec{r}_3) n(\vec{r}_3; t) \cdot \frac{\partial}{\partial \vec{v}_1} + \int_{\mathbb{R}^3} d^3 r_3 \vec{F}(\vec{r}_2 - \vec{r}_3) n(\vec{r}_3; t) \cdot \frac{\partial}{\partial \vec{v}_2} \right\} g_2(\mathbf{1}, \mathbf{2}; t) \quad (24)$$

where $n(\vec{r}; t)$ is the local number density defined in Chapter 2. The integral $\vec{\mathcal{F}}(\vec{r}_1) \equiv \int d^3r_3 \vec{F}(\vec{r}_1 - \vec{r}_3) n(\vec{r}_3; t)$, the Vlasov mean force field, represents N times the statistical mean of the force that another particle 3 exerts on particle 1 averaged on the position probability density $p(\vec{r}_3; t) \equiv \frac{n(\vec{r}_3; t)}{N}$. The second integral has the same meaning but with particle 1 replaced by particle 2. Let us, then, compare K_4 to the term \mathcal{L} written more explicitly as

$$\mathcal{L} = -\frac{1}{m} \vec{F}(\vec{r}_1 - \vec{r}_2) \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) [g(\mathbf{1}, \mathbf{2}; t) + f(\mathbf{1}; t) f(\mathbf{2}; t)] \quad (25)$$

We must compare the orders of magnitude of $\|\vec{F}(\vec{r}_1 - \vec{r}_2)\|$ and $\|\vec{\mathcal{F}}(\vec{r}_i)\|$, $i = 1, 2$, for $\|\vec{r}_2 - \vec{r}_1\| < d$. One has $\|\vec{F}(\vec{r}_1 - \vec{r}_2)\| > \frac{\gamma}{d^2}$. As to $\vec{\mathcal{F}}(\vec{r}_i)$, using integration by part, it can be rewritten as $\vec{\mathcal{F}}(\vec{r}_i) \equiv -\int d^3r_3 U(\vec{r}_i - \vec{r}_3) \frac{\partial}{\partial \vec{r}_3} n(\vec{r}_3; t)$ where we used the fact that the potential $U(\vec{r}) \rightarrow 0$ for $r \rightarrow \infty$ and $n(\vec{r}; t) \rightarrow n = N/V$ for $r \rightarrow \infty$. Clearly, $\vec{\mathcal{F}}(\vec{r}_1)$ vanishes for homogeneous systems. The integrand in $\vec{\mathcal{F}}(\vec{r}_i)$ is vanishingly small in every part of the integration domain where the gradient of the local number density, $\|\frac{\partial}{\partial \vec{r}} n(\vec{r}; t)\|$, is vanishingly small. Let us call L_H the typical length on which $n(\vec{r}; t)$ varies noticeably. Thus, the volume of integration in which the integrand does not vanish is of order L_H^3 . Consequently, the order of magnitude of the integral defining $\|\vec{\mathcal{F}}(\vec{r}_i)\|$ is $\frac{\gamma}{L_H} \cdot \frac{n}{L_H} \cdot L_H^3 = \gamma n L_H$. Hence, K_4 is negligible with respect to \mathcal{L} if $\frac{\gamma}{d^2} > \gamma n L_H$. This inequality transforms into $\Gamma^2 < \frac{\delta}{L_H} \ll 1$ which is compatible with the physical conditions formulated in Chapter 2.

The free motion term $[L_1^0 + L_2^0] g_2(\mathbf{1}, \mathbf{2}; t)$ of equation (6) is also negligible compared to the term \mathcal{L} . The latter is proportional to the force between particles 1 and 2 which is of order Γ^{-2} . Indeed, one has $\|\vec{F}(\vec{r}_1 - \vec{r}_2)\| > \frac{\gamma}{d^2} = \Gamma^{-2} \gamma n^{2/3}$ while the free motion operator $[L_1^0 + L_2^0]$ is independent of Γ .

With these arguments, what remains from equation (6) is equation (7).

2. Establishing the kinetic equation (10)

The equation (7) is solved by adding the solution of the homogeneous part of this equation to the convolution of the propagator of the homogeneous equation with the source term $L'_{12}f(\mathbf{1};t)f(\mathbf{2};t)$. Using the Fourier-transform with respect to the velocities and some simple algebra, one gets

$$\begin{aligned} \tilde{g}_2(\vec{r}_1, \vec{\zeta}_1, \vec{r}_2, \vec{\zeta}_2; t) &= \tilde{U}(t) \tilde{g}_2(\vec{r}_1, \vec{\zeta}_1, \vec{r}_2, \vec{\zeta}_2; 0) \\ &+ \frac{\partial}{\partial \alpha} \int_0^t \frac{d\tau}{\tau} \tilde{U}(\alpha\tau) \tilde{f}(\vec{r}_1, \vec{\zeta}_1; t - \tau) \tilde{f}(\vec{r}_2, \vec{\zeta}_2; t - \tau) |_{\alpha=1} \end{aligned} \quad (26)$$

where $\vec{\zeta}_1$ and $\vec{\zeta}_2$ are the Fourier variables associated to the velocities \vec{v}_1 and \vec{v}_2 and where

$$\tilde{U}(t) = \exp\left[\left(-\frac{i}{m} \vec{F}_{12} \cdot (\vec{\zeta}_1 - \vec{\zeta}_2)\right)t\right] \quad (27)$$

is the Fourier-transform of the propagator of the homogeneous part of equation (7).

From here on, vanishing initial correlation $g_2(\vec{r}_1, \vec{v}_1; \vec{r}_2, \vec{v}_2; 0)$ is assumed. As discussed in chapter 5, non-vanishing initial correlation would not change our main conclusions.

Taking the inverse Fourier transform of expression (26), introducing the resulting formula for $g_2(\mathbf{1}, \mathbf{2}; t)$ in the formula (3) of I_1 and after a permutation of integrals, one obtains

$$\begin{aligned} I_1 &= \frac{\partial}{\partial \alpha} \int d^3v_2 \int \frac{d^3\zeta_1 d^3\zeta_2}{(2\pi)^6} e^{i\vec{\zeta}_1 \cdot \vec{v}_1 + i\vec{\zeta}_2 \cdot \vec{v}_2} \\ &\int_0^t \frac{d\tau}{\tau} \int_{S_1} d^3r_2 \frac{(-i)}{m} \vec{F}_{12} \cdot (\vec{\zeta}_1 - \vec{\zeta}_2) e^{-\frac{i\alpha}{m} \vec{F}_{12} \cdot (\vec{\zeta}_1 - \vec{\zeta}_2)\tau} \tilde{f}(\vec{r}_1, \vec{\zeta}_1; t - \tau) \tilde{f}(\vec{r}_2, \vec{\zeta}_2; t - \tau) |_{\alpha=1} \end{aligned} \quad (28)$$

Since $d \ll L_H$, one can approximate $\tilde{f}(\vec{r}_2, \vec{\zeta}_2; t - \tau)$ by its value at $\vec{r}_2 = \vec{r}_1$ and extract it from the integral over \vec{r}_2 in the ball S_1 . With some algebra, equation (28)

becomes

$$I_1 \approx -\frac{\partial^2}{\partial \alpha^2} \int \frac{d^3 \zeta_1}{(2\pi)^3} e^{i\vec{\zeta}_1 \cdot \vec{v}_1} \int_0^t \frac{d\tau}{\tau^2} \tilde{f}(\vec{r}_1, \vec{\zeta}_1; t - \tau) n(\vec{r}_1; t - \tau) J|_{\alpha=1} \quad (29)$$

with

$$J \equiv \int_{S_1} d^3 r \left(e^{-\frac{i\alpha}{m} \vec{F}(\vec{r}) \cdot \vec{\zeta}_1 \tau} - 1 \right) \quad (30)$$

After a change of variable $\vec{r} \rightarrow \vec{F}(\vec{r})$ and passing to spherical coordinates, J transforms into

$$J = -2\pi \left(\frac{\gamma \zeta_1 \alpha \tau}{m} \right)^{3/2} \left(\frac{2}{3} (z_m)^{-3/2} - \int_{z_m}^{\infty} dz z^{-7/2} \sin z \right) \quad (31)$$

where $z_m \equiv \frac{\gamma \alpha \tau \zeta_1}{d^2 m}$. The above integral is an incomplete Sine-integral function whose power-series expansion in z_m (see [19]) leads to

$$J = 2\pi d^3 \left[\frac{4}{15} \sqrt{2\pi} \left(\frac{\gamma \alpha \tau \zeta_1}{d^2 m} \right)^{3/2} - \frac{1}{3} \left(\frac{\gamma \alpha \tau \zeta_1}{d^2 m} \right)^2 + \frac{1}{300} \left(\frac{\gamma \alpha \tau \zeta_1}{d^2 m} \right)^4 + \frac{1}{3740} \left(\frac{\gamma \alpha \tau \zeta_1}{d^2 m} \right)^6 + \dots \right] \quad (32)$$

We, now, can discuss the question raised in Chapter 2 (after equation (5)) about the convergence of the integral I_1 . In its form (29), the only place where the force $\vec{F}(\vec{r})$ appears is the integral J given by equation (30). As seen from its result (32), J converges. This comes from the fact that the force appears only in a phase factor in equation (30).

Coherently with our short-time assumption, we suppose $z_m \ll 1$ and retain only the first term of the series. The upper boundary t of the time integral in equation (29), thus, must be such that $\frac{\gamma \alpha t \zeta_1}{d^2 m} \ll 1$. More explicitly, let us replace ζ_1 by the inverse of an average velocity v_{av} and put $\alpha = 1$. This transforms the previous inequality into $\frac{\gamma}{d^2 m} t \ll v_{av}$. In other words, the time t must be such that the velocity increment $\Delta v = \frac{\gamma}{d^2 m} t$ acquired by particle 1 during time t under the force of another particle at the surface of S_1 , satisfies $\Delta v \ll v_{av}$.

Finally, introducing the first term of series (32) in equation (29), one gets equation (8) which, in turn, leads to the kinetic equation (10) in Chapter 2.

B. Solution of the equation 11

A Fourier transform with respect to \vec{v} and a Laplace transform with respect to t of equation (11) give

$$\hat{\tilde{\varphi}}(\vec{\zeta}; w) = \frac{w^{1/2} \tilde{\varphi}(\vec{\zeta}; 0)}{w^{3/2} + A |\vec{\zeta}|^{3/2}} \quad (33)$$

where $\tilde{\varphi}(\vec{\zeta}; 0)$ is the Fourier transform of $\varphi(\vec{v}, t)$ at $t = 0$, $\hat{\tilde{\varphi}}(\vec{\zeta}; w)$ is the Fourier-Laplace transform of $\varphi(\vec{v}, t)$, and $A = \frac{n\sqrt{\pi}}{5} \left(\frac{2\pi\gamma}{m}\right)^{3/2}$. The inverse Laplace transform of $\hat{\tilde{\varphi}}(\vec{\zeta}; w)$ is taken by first expanding equation (33) in powers of $A |\vec{\zeta}|^{3/2}$ and, then, integrating the series term by term. This leads to the exact solution of equation (11)

$$\varphi(\vec{v}; t) = \int \frac{d^3\zeta}{(2\pi)^3} e^{i\vec{\zeta}\cdot\vec{v}} \tilde{\varphi}(\vec{\zeta}; 0) E_{3/2}(-A \zeta^{3/2} t^{3/2}) \quad (34)$$

where

$$E_\mu(u) = \sum_{k=0}^{\infty} \frac{(u)^k}{\Gamma(\mu k + 1)} \quad (35)$$

is the Mittag-Leffler function of parameter μ [15]. The condition $z_m \ll 1$ assumed in our derivation of the kinetic equation is obviously compatible with condition $A \zeta^{3/2} t^{3/2} \ll 1$. We, thus, can safely make the following approximation

$$E_{3/2}(-A \zeta^{3/2} t^{3/2}) \simeq e^{-C(\zeta t)^{3/2}} \quad (36)$$

with $C = \frac{4n}{15} \left(\frac{2\pi\gamma}{m}\right)^{3/2}$. Finally, we get

$$\varphi(\vec{v}; t) \simeq \int \frac{d^3\zeta}{(2\pi)^3} e^{i\vec{\zeta}\cdot\vec{v}} \tilde{\varphi}(\vec{\zeta}; 0) e^{-C(\zeta t)^{3/2}} \quad (37)$$

equivalent to the velocity convolution of the initial velocity distribution and a Lévy-3/2 distribution [2]

$$\varphi(\vec{v}; t) \simeq \int d^3u \varphi(\vec{u}; 0) L_{3/2}(\vec{v} - \vec{u}, Ct^{3/2}) \quad (38)$$

Using a theorem in reference [7] one then shows that for any $\varphi(\vec{v}, 0)$ with finite second moments, this approximation as well as the exact solution (34) have a long tail in $1/v^{5/2}$ where v is any component of the velocity vector \vec{v} .

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